Critical points in two-dimensional replica sigma models

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Abstract

I survey the kinds of critical behavior believed to be exhibited in two-dimensional disordered systems. I review the different replica sigma models used to describe the low-energy physics, and discuss how critical points appear because of WZW and theta terms.

The last few years have seen a remarkable resurgence in activity on disordered systems in two dimensions. Even though serious study in this field goes back more than twenty years, recently there have been a number of precise quantitative results. The most famous experimental problem of this sort is the transition between plateaus in the quantum Hall effect. The experiments suggest strongly that there is a critical point in between the plateaus, and all the theoretical explanations of the quantum Hall effect indicate that disorder plays a crucial role in this phase transition.

In this talk, I will discuss the kinds of critical behavior which happen in systems of electrons which interact only with disorder. It is possible that electron interactions will substantially affect the problem. However, until we theoretically understand the problem without interactions, it is of course unlikely that we will be able to understand the role of electron interactions.

In [1, 2] Hamiltonians bilinear in electron annihilation/creation operators are classified. The disorder appears in the couplings of the Hamiltonian, i.e. the coefficients of the various terms are taken to be random with some distribution. For a system with a finite number of states, each Hamiltonian is some matrix belonging to the appropriate symmetry class (e.g. Hermitian, real symmetric, etc.). The simplest (zero-dimensional) way to analyze these Hamiltonians would be to consider an ensemble of random matri-

ces belonging to the appropriate symmetry class. It turns out these classes are related to symmetric spaces, so the random matrix ensembles are conveniently labelled by a corresponding symmetric space [1, 2]. I present these labels in the first table, with a brief physical description of the corresponding systems. For example, models in the first three classes have a Hamiltonian describing electrons with spin hopping on a lattice, with or without $SU(2)$ spin symmetry $\mathcal{S}$ and a discrete time-reversal symmetry $\mathcal{T}$. Classes AIII and CII describe electrons with no spin, but their Hamiltonians allow Cooper pairing terms which simultaneously annihilate or create a pair of electrons. The final four Hamiltonians have spin and Cooper pairing terms. The descriptions in parentheses will be discussed later in this talk.

RMT	description
A (GUE)	Anderson localization, broken $\mathcal{S},\mathcal{T}$ (Hall plateau)
AI (GSE)	Anderson localization, broken $\mathcal{S}$, good $\mathcal{T}$
AII (GOE)	Anderson localization, good $\mathcal{S},\mathcal{T}$
AIII	spinless superconductor, broken $\mathcal{T}$
BDI	
CII	spinless superconductor, good $\mathcal{T}$
C	superconductor, broken $\mathcal{T}$, good $\mathcal{S}$ (SQHE, $d_{x^2-y^2}+id_{xy}$)
CI	superconductor, good $\mathcal{T},\mathcal{S}$ ($d_{x^2-y^2}$)
D	superconductor, broken $\mathcal{T},\mathcal{S}$ (random-bond Ising)
DIII	superconductor, good $\mathcal{T}$, broken $\mathcal{S}$

Table 1: Random matrix theories and some physical descriptions

It is worth noting that properties at non-zero frequency $\omega \neq 0$ are described by the GOE, GSE or GUE universality classes; the other universality classes only describe $\omega \to 0$ properties.

Since here I am interested in these systems in two spatial dimensions, I will not utilize random matrix theory. However, because classification according to random matrix ensembles is very convenient for these sorts of Hamiltonians, most workers in this field still refer to the systems by the names in the first column of table 1.

The disordered systems in table I can be described by certain sigma models in the replica formulation. I review here what these words mean.

Disorder is introduced into these fermion systems by allowing their couplings to vary randomly with some distribution. The action is $S(\psi, R)$, where the fermions are denoted ψ and the random variables R. The type of disorder considered here is called quenched, meaning physical quantity is computed first for fixed disorder, and then the randomness

is averaged over. For example, the free energy at fixed disorder $\ln Z(R)$ follows from the path integral over the fermions for a given configuration R:

$$\ln Z(R) = \ln \left[\int [\mathcal{D}\psi] e^{-S(\psi,R)} \right].$$

The physical free energy $\bar{f}$ is then given by averaging over the random variables R:

$$\bar{f} = \int [\mathcal{D}R] \ln Z(R). \tag{1}$$

The replica trick is a method for computing physical quantities which seems to work at least some of the time. For the free energy, the replica trick uses the identity

$$\ln Z = \lim_{N\to 0} \frac{Z^N - 1}{N}$$

to rewrite $\ln Z(R)$ in (1). The partition function to the N^{th} power can be written as the product of N path integrals over N "replica" fields ψ_μ where $\mu = 1 \dots N$, but all with the same random fields R. Then one makes the bold assumption that the $N \to 0$ limit and the $N+1$ path integrals commute, giving

$$\bar{f} = \lim_{N\to 0} \frac{1}{N} \left[-1 + \int [\mathcal{D}R] \int [\mathcal{D}\psi_1] e^{-S(\psi_1,R)} \int [\mathcal{D}\psi_2] e^{-S(\psi_2,R)} \dots \int [\mathcal{D}\psi_N] e^{-S(\psi_N,R)} \right].$$

This trick thus puts the random fields on the same footing as the replica fields, and one can use standard field theory techniques on the problem.

The catch is that at the end of the computation one needs to set the number of fields N to be zero. Since rarely in a field theory does one understand the analyticity properties of N, this step is certainly not rigorous, or even close. Below I will take the strategy of just doing computations at positive integer N where everything is well defined, and hope that the formulas are still valid when N is set to be zero. This procedure is well defined perturbatively, but we will attempt to extract non-perturbative information as well. There is another approach to these problems which utilizes supergroup symmetries instead of the replica trick. This approach does not suffer from all the possible ambiguities in taking this $N \to 0$ limit. However, it is not known how to utilize the techniques of integrability in the supergroup formulation, so I am stuck with the replica trick.

To make further progress on the problem we map the problem onto a sigma model [3]. The sigma model is an effective field theory which follows (uniquely) by analyzing the symmetries of the replica field theory. A sigma model is a field theory where the fields take values on some manifold. In this talk, all of these manifolds are of the form G/H, where G and H are Lie groups, with H a maximal subgroup of G. Such a manifold is called a symmetric space. The best-known example of a sigma model on a symmetric space is often called the sphere sigma model, where $G = O(3)$ and $H = O(2)$. The field can be thought of as a vector with three components and fixed length, hence a

sphere. While the vector can be rotated by the $O(3)$ group, it is invariant under the $O(2)$ subgroup consisting of rotations around its axis. Thus the space of distinct three-dimensional fixed-length vectors (the two-sphere) is the coset $O(3)/O(2)$.

The replica sigma models describing the models in Table 1 can read off the tables of [2]. To see where they come from, it is useful to do one example in detail. This discussion I steal from [4]. The Hamiltonian describes spinless fermions with a triplet p-wave type pairing:

$$H_0 = \sum_k \epsilon_k c_k^\dagger c_k + (\Delta_k c_k^\dagger c_{-k}^\dagger + \text{h.c.}) \tag{2}$$

where $\Delta_k \sim \frac{v_\Delta}{2} k_x$. Disorder is weak enough to maintain some notion of the Fermi surface. The low energy excitations of the fermions are then found about two nodes positioned on the k_y-axis at $K_\pm = (0, \pm K)$. Linearizing the theory about these nodes via $\epsilon_{K_\pm + q} = q_y v_F$ and $c \sim c_1 \exp(iK_+ x) + c_2 \exp(iK_- x)$, H becomes

$$H = \int d^2x\ \psi^\dagger (iv_F \tau_z \partial_y + iv_\Delta \tau_x \partial_x)\psi. \tag{3}$$

where $\psi^\dagger = (c_1^\dagger, c_2)$. The Pauli matrices τ_i act in the particle-hole space of the spinors. H is precisely the Hamiltonian of a single Dirac fermion in $2+1$ dimensions. One can compute correlators at fixed frequency ω by utilizing the action of a classical two-dimensional Euclidean field theory:

$$S_0 = \int d^2x\ \psi^\dagger (iv_F \tau_z \partial_y + iv_\Delta \tau_x \partial_x - i\omega\tau_z)\psi. \tag{4}$$

One way of adding randomness is to include the term

$$H_{\text{disorder}} = R(c_1^\dagger c_1 + c_2^\dagger c_2) \tag{5}$$

in the Hamiltonian. Here R is a random variable with variance $\langle R(x)R(y)\rangle = u^{-1}\delta(x-y)$; because of the delta function the disorder is called on-site. After replicating the fermions, $\psi \to \psi_k$, the path integral over the random field R is easily done. This couples the different replicas via the quartic terms $S_{\text{disorder}} = -\frac{1}{u}(\psi_k^\dagger \tau^z \psi_k)(\psi_l^\dagger \tau^z \psi_l)$. Reorganizing the fields via $\tilde{\psi}^\dagger \equiv (\psi_R^\dagger, \psi_L^\dagger) = \psi^\dagger \tau^z \exp(i\pi\tau^x/4)$ and $\tilde{\psi} \equiv (\psi_R, \psi_L) = \exp(-i\pi\tau^x/4)\psi$ gives the action

$$S = \int d^2x v_F \tilde{\psi}_k^\dagger (i\partial_y - \tau^z \partial_x)\tilde{\psi}_k - \frac{1}{u}(\tilde{\psi}_k^\dagger \tilde{\psi}_k)(\tilde{\psi}_l^\dagger \tilde{\psi}_l). \tag{6}$$

This theory is invariant under the group $U(N)_L \times U(N)_R$ transforming $\tilde{\psi} \to \frac{1}{2}((1+\tau^z)U_L + (1-\tau^z)U_R)\tilde{\psi}$. Adding disorder to the Cooper-pairing term results in another four-fermion term, but preserves this symmetry. As long as the symmetry structure is unchanged, the low-energy physics should be the same.

The sigma model describes the low-energy behavior of this system. The idea is familiar from many different contexts in condensed-matter and particle physics. The

theory has a global symmetry G, which for this example is $U(N) \times U(N)$. However, we assume there is an energy scale where some fermion bilinear gets an expectation value. This expectation value is invariant under only a subgroup H of the full symmetry group G. In dimensions higher than two, this would result in spontaneous symmetry breaking. The sigma model describes the interactions of the resulting Goldstone bosons, which take values on the space G/H. In two dimensions, the symmetry does not break spontaneously (G remains a good global symmetry of the low-energy theory), but still the low-energy degrees of freedom live on G/H. The easiest way to see this explicitly is to introduce a Hubbard-Stratonovich matrix field M_{kl} to factor the four-fermion term:

$$S = \int d^2x \left[iv_F \tilde{\psi}_k^\dagger (\partial_y - \tau^z \partial_x) \tilde{\psi}_k - \frac{1}{u} (\tilde{\psi}_k^\dagger M_{kl} \tilde{\psi}_l) + \mathrm{tr}\, M^2 \right] \tag{7}$$

where M is hermitian. Under the symmetry $U(N)_L \times U(N)_R$, $M \to UMU^\dagger$. The sigma model describes the physics around saddle points of this path integral. For example, one can have saddle points where M is off-diagonal, e.g. $M_{LL} = M_{RR} = 0$, but $M_{LR} = M_{RL}^\dagger \propto I$, the identity. The diagonal subgroup $U(N)_V$ leaves this saddle point invariant, and so the low-energy modes $T = M_{RL}$ take values in $U(N)_L \times U(N)_R / U(N)_V \approx U(N)$. Focusing solely upon these modes gives

$$S = S_0 - \frac{1}{u} (\tilde{\psi}_R^\dagger T \tilde{\psi}_L + \tilde{\psi}_L^\dagger T^\dagger \tilde{\psi}_R). \tag{8}$$

Integrating out the fermions leaves an effective action for the bosonic field T, which can be expanded in in powers of the momentum over the expectation value of T. I will discuss the form of this action later. The important thing is that it follows solely from the symmetry.

This model is in class AIII in Table 1. This result can be read off from the tables in [2]. The replica sigma model corresponding to a given disordered universality class is determined by taking G/H to be the bosonic subspace labeled "M_F" in the second table in [2]. The F is for fermion; we are using fermionic replicas here. If we instead had used bosonic replicas to compute the same Green's functions, we would have ended up with a replica sigma model on a non-compact space; this model is labelled M_B in Zirnbauer's table. For the above example, this would be $Gl(N, C)/U(N)$. The non-compact sigma model is not equivalent to the compact one except hopefully in the replica limit $N \to 0$. However, very few exact results are available for sigma models on non-compact spaces (in fact, the difficulty in obtaining exact results for the supergroup sigma models arises mainly from the non-compact bosonic subgroup, not from the fermionic fields).

I will concentrate here exclusively on the replica sigma models on compact spaces. I present a two-dimensional version of Zirnbauer's tables in Table 2. The first column is the commonly-used label coming from figure 1; I have rearranged the rows for reasons which will become clear later. The second column is the replica sigma model. The third column lists the types of critical behavior possible in this sigma model. It is the purpose of the rest of this paper to discuss the third column.

RMT	replica sigma model	possible 2D critical behavior
GUE	$U(2N)/U(N) \times U(N)$	Pruisken phase
C	$Sp(2N)/U(N)$	Pruisken phase
D	$O(2N)/U(N)$	Pruisken phase, metallic phase
CII	$U(N)/O(N)$	$\theta = \pi \to U(N)_1$; Gade phase
GSE	$O(2N)/O(N) \times O(N)$	$\theta = \pi \to O(2N)_1$; metallic phase
AIII	$U(N) \times U(N)/U(N)$	WZW term; Gade phase
CI	$Sp(2N) \times Sp(2N)/Sp(2N)$	WZW term
DIII	$O(N) \times O(N)/O(N)$	WZW term; metallic phase
BDI	$U(2N)/Sp(2N)$	Gade phase?
GOE	$Sp(4N)/Sp(2N) \times Sp(2N)$	none!

Table 2: The replica sigma models and their possible critical behavior

The sphere sigma model we have discussed is the $N = 1$ case of the classes GUE, C, CII, and the N=2 case of classes D and GSE (the latter comprises two copies of the sphere).

It is important to note that even though each random matrix theory is associated with a symmetric space as discussed in [2] (this is the origin of most of the names), this symmetric space is **not** necessarily the same as the symmetric space of the corresponding replica sigma model. For example, the random matrices for the theory CI are in the tangent space of $Sp(2N)/U(N)$, (i.e. exponentiating the random matrices gives the space CI$\equiv$ $Sp(2N)/U(N)$), but the corresponding replica sigma model is on the symmetric space $Sp(2N) \times Sp(2N)/Sp(2N)$.

Sigma models on symmetric spaces have the convenient property that they have only one coupling constant. Roughly speaking, this means that they preserve their "shape" under renormalization: only their size changes. For example, the sphere sigma model remains a sphere under renormalization: only the radius of the sphere renormalizes. Actually, the sigma models for AIII, CII and BDI are not quite symmetric spaces: the corresponding symmetric spaces have $SU(N)$ instead of $U(N)$. The extra $U(1)$ has some interesting effects (for example turning a critical point into a critical line), but does not affect the sigma model on the symmetric space at all.

At low energy, one can safely neglect four-derivative terms and higher in the action. All the above "ordinary" sigma models have an action can be written in the form

$$S_{ordinary} = g \int dx\, dy\ \mathrm{tr} \left[\partial_\mu T\, \partial^\mu T^{-1}\right] \tag{9}$$

where T is a unitary matrix, possibly with further restrictions. This action is very non-trivial because of the non-linear constraint of unitarity. For the example above, T is an $N \times N$ unitary matrix, because $U(N) \times U(N)/U(N) \approx U(N)$ as a space. For T in $U(N)/O(N)$ (class CII), T is an $N \times N$ symmetric unitary matrix. Symmetric matrices do not close under multiplication, which is why $U(N)/O(N)$ is not a group but rather a

coset. For $O(2N)/O(N) \times O(N)$ (class GSE), T is a $2N \times 2N$ symmetric real orthogonal matrix.

An important observation of Anderson's is that the coupling constant g of the sigma model is related to the conductance of the system: if g renormalizes to zero, the system is localized and not conducting, while if g renormalizes to infinity, the system is metallic. The latter is the trivial fixed point of the system. The latter corresponds to the size of the manifold (e.g. the radius of the sphere) going to infinity , which means the manifold becomes effectively flat. For ordinary sigma models on compact symmetric spaces with $N > 1$, the trivial fixed point is unstable. The beta function is proportional to the curvature of the manifold [5], which is always positive for these spaces. Moreover (for $N > 1$), the beta function shows no evidence for a non-trivial fixed point. This is a more or less a consequence of the Mermin-Wagner-Coleman theorem, which says that continuous symmetries can not be spontaneously broken in two dimensions. This implies that the manifold should be always curved (and not critical); otherwise there would be Goldstone modes in violation of the theorem.

However, in two dimensions, there are a number of different ways critical behavior appears in a disordered system. I group them into three types:

1. Perturbative Peculiarities
   a) Gade phase
   b) Metallic phase
2. WZW term
3. Theta term
   a) Pruisken type (non-vanishing σ_{xy})
   b) $\theta = 0$ or π only

We will discuss all these methods, but devote most of our attention to 2 and 3. These require adding extra terms to the action $S_{ordinary}$.

1. **Perturbative Peculiarities**

While it is important to note that the Mermin-Wagner-Coleman theorem implies no non-trivial fixed points in ordinary sigma models for $N > 1$, this of course does not prohibit interesting things from happening in the replica limit $N \to 0$ [6]. For example, the dimension of all the manifolds in table II goes to zero as $N \to 0$, making the idea of positive curvature somewhat confusing. In fact, a while ago (see [7] and references within) it was shown that at one loop

$$\begin{aligned}
\beta &\propto N && \text{if } G = U(N) \\
\beta &\propto N - 2 && \text{if } G = O(N) \\
\beta &\propto N + 1 && \text{if } G = Sp(2N)
\end{aligned}$$

where the constant of proportionality is a negative number for $N \geq 0$. Therefore, by "perturbative peculiarities" I mean the consequences of the fact that as $N \to 0$ the β function goes to zero for $G = U(N)$ and has changed sign for $G = O(N)$.

If the beta function goes to zero to all orders as $N \to 0$, this opens up the possibility that there is no flow in the sigma model: for any g the model is at a fixed point. This indeed happens for classes AIII and CII (and probably BDI, although I am not aware of any explicit computations other than of the perturbative beta function). This behavior was discussed at length in [8], which is why I call this the Gade phase. In Gade's work these universality classes are realized by a particle hopping on a bipartite lattice (the particle is restricted to hop only from one sublattice to the other); class CII has time-reversal symmetry, while class AIII does not. The supergroup approach to these models was discussed in detail in [9]. Since these models also have an extra $U(1)$ factor as mentioned above, the model is critical over an entire plane of couplings.

If the beta function is positive as $N \to 0$, the trivial fixed point is stable. This happens for classes D, DIII and GSE. Since g renormalizes to infinity, this phase is conducting and is hence called metallic. This implies the existence of at least one non-trivial fixed point, because it is still expected that for strong enough disorder, the system does not conduct. Hence there should be a phase transition from the metallic phase to a localized phase at some value of g. This fixed point should be unstable in g in both directions (i.e. for $g < g_c$ the system renormalizes to $g \to 0$, while for $g > g_c$ the system renormalizes to $g \to \infty$).

2. **WZW term**

Two-dimensional sigma models with a manifold of the form $H \times H/H$ are called principal chiral models, and have been widely studied. They are massive asymptotically-free field theories for $N > 1$. However, there is an additional term which can be added to the action (9) which changes the low-energy behavior from gapped to gapless for any N. This is called the Wess-Zumino-Witten term. To write it out explicitly, first one needs to consider field configurations $h(x, y)$ which fall off at spatial infinity, so that one can take the spatial coordinates x and y to be on a sphere. Then one needs to extend the fields $h(x, y)$ on the sphere to fields $h(x, y, z)$ on a ball which has the original sphere as a boundary. The fields inside the ball are defined so that h at the origin is the identity matrix, while h on the boundary is the original $h(x, y)$. It is possible to find a continuous deformation of $h(x, y)$ to the identity because $\pi_2(H) = 0$ for any simple Lie group. Then the WZW term is $k\Gamma$, where

$$\Gamma = \frac{\epsilon_{abc}}{24\pi} \int dxdydz \, \mathrm{tr} \left[(h^{-1}\partial_a h)(h^{-1}\partial_b h)(h^{-1}\partial_c h) \right]. \tag{10}$$

The coefficient k is known as the level, and for compact groups must be an integer because the different possible extensions of $h(x, y)$ to the ball yield a possible ambiguity of $2n\pi$ in Γ. The WZW term changes the equations of motion and beta function, but only by terms involving $h(x, y)$: the variation of the integrand is a total derivative in z.

The two-dimensional sigma model with WZW term has a stable fixed point at $1/g = 16\pi/k$ [10], so the model is critical and the quasiparticles are gapless. The corresponding conformal field theory is known as the H_k WZW model [11]. The WZW term is invariant under discrete parity transformations (e.g. $x \to -x$, $y \to y$) if $h \to h^{-1}$ under this transformation. For there to be a WZW term in a parity-invariant theory, some of the low-energy fields must be pseudoscalars.

The WZW term allows some of these sigma models which has a stable low-energy fixed point. Why this term must arise in many situations was understood in particle physics some time ago. This was the topic of [4]. The WZW term was shown to arise in several disordered systems [12, 13], by bosonizing the explicit system. There it appears to ensure the certain current-algebra commutation relations are obeyed properly in the bosonic theory. It appears more generally in the map of the action like (7) to a sigma model. The sigma model arises when integrating out the fermions in this action, leaving a low-energy effective action for h. Upon doing so, one easily obtains the ordinary sigma model action of the form (9) (with $T = h$). The WZW term arises for a subtler reason. To perform a consistent low-energy expansion of the action , one must change field variables. This results in a Jacobian in the path integral [14], which in these two-dimensional cases is precisely the WZW term (10).

In fact, one can determine without these explicit computations whether or not the WZW term will appear in the low-energy effective action. The reason is some very deep physics known as the chiral anomaly. In the models which admit a WZW term, the fermions have a chiral $H_L \times H_R$ symmetry. As is well known, chiral symmetries involving fermions are frequently anomalous. Noether's theorem says that a symmetry of the action gives a conserved current with $\partial_\mu j^\mu = 0$, but this is only true to lowest order in perturbation theory. An anomaly is when the current is not conserved in the full theory (although the associated charge is still conserved). In the case of massless fermions in $1+1$ dimensions, this was shown in detail in [15, 10]. The WZW term is the effect of the anomaly on the low-energy theory. Even though the fermions effectively become massive when T gets an expectation value, their presence still has an effect on the low-energy theory, even if this mass is arbitrarily large. This violation of decoupling happens because the chiral anomaly must be present in the low-energy theory. In other words, the anomaly coefficient does not renormalize. This follows from an argument known as 't Hooft anomaly matching [16]. One imagines weakly gauging the anomalous symmetry. It is not possible to gauge an anomalous symmetry in a renormalizable theory, but one can add otherwise non-interacting massless chiral fermions to cancel the anomaly. Adding these spectator fermions ensures that the appropriate Ward identities are obeyed, and the symmetry can be gauged. In the low-energy effective theory, the Ward identities must still be obeyed. Because the massless spectator fermions are still present in the low-energy theory, there must be a term in the low-energy action which cancels the anomaly from the spectators. This is the WZW term.

To determine whether a chiral anomaly and hence a WZW term is present, one usually needs to do only a simple one-loop computation. For chiral symmetries, it is

customary to define the vector and axial currents $j^\mu_V = j^\mu_L + j^\mu_R$, $j^\mu_A = j^\mu_L - j^\mu_R$; for these theories $j^\mu_A = \epsilon^{\mu\nu} j_{V\nu}$. One then computes the correlation functions $\langle j^\mu_V(x,y) j^\nu_V(0,0)\rangle \equiv C^{\mu\nu}$. If $\partial_\mu C^{\mu\nu} \neq 0$ or $\epsilon_{\alpha\beta}\partial^\alpha C^{\beta\nu} \neq 0$ for some ν, then there is an anomaly. An important characteristic of the anomaly is that it is independent of any continuous change in the theory, as long as the chiral symmetry is not broken explicitly. Thus to find the anomaly, one can compute $C^{\mu\nu}$ using free fermions (where the only contribution is a simple one-loop graph, see e.g. [10]).

For the p-wave superconductor example discussed in detail above (class AIII), one has $H = U(N)$ and the possibility of a WZW term. With two nodes in the fermi surface as discussed above, one in fact does, with $k = 1$ [17, 4]. The model is already critical because it is in a Gade phase, but the WZW term does have an effect; for example when the coupling g is at the WZW fixed point, the density of states falls off with a power law in energy instead of an exponential [9].

Classes CI and DIII are more interesting in this context.

Class CI basically amounts to generalizing (2) to include the spin of the electron, and preserving the $SU(2)$ spin symmetry [18]. As discussed in [12, 4], if there are two nodes in the fermi surface, one ends up with an $Sp(2N)_1 = U(N)_2$ critical point. The case of most interest is the $d_{x^2-y^2}$ superconductor, where there are four nodes in the Fermi surface. If the two sets of nodes are not coupled then the global symmetry is enlarged to $Sp(2N) \times Sp(2N) \times Sp(2N) \times Sp(2N)$. 't Hooft's argument means that one obtains two copies of the $Sp(2N)_1$ theory with conserved currents

$$\partial_\mu j^\mu_{V1} = \partial_\mu j^\mu_{V2} = 0$$

$$\partial_\mu j^\mu_{A1} = \partial_\mu j^\mu_{A2} \neq 0.$$

The reason the anomalies in the second line are the same is that I have defined the two sets of fields with the same conventions (i.e. the same sign WZW term). However, generic disorder couples all the nodes, breaking the symmetry back to $Sp(2N) \times Sp(2N)$. If the remaining symmetries are generated by $j^\mu_{V1} + j^\mu_{V2}$ and $j^\mu_{A1} + j^\mu_{A2}$, then the latter is anomalous. Because of 't Hooft's argument, the two anomalies just add and one obtains $Sp(2N)_2$ [4]. However, the conserved symmetries of the $d_{x^2-y^2}$ superconductor are determined by the band structure, and this requires that the conserved currents with these conventions be $j^\mu_{V1} + j^\mu_{V2}$ and $j^\mu_{A1} - j^\mu_{A2}$ [19]. The anomalies in the latter cancel, and therefore there is no WZW term and no phase transition in the resulting sigma model. I have used relativistic notation in this section, but even if 1+1 dimensional Lorentz invariance is broken (e.g. by taking $v_{F1}/v_{\Delta 1} \neq v_{F2}/v_{\Delta 2}$), these results still apply.

A WZW term likewise appears in class DIII when there are two nodes. For $N > 2$ this fixed point is stable like the others. As discussed above, the sign of the beta function has flipped as $N \to 0$, so the model has a metallic phase. This makes the $N \to 0$ limit of the $O(2N)_k$ fixed point an excellent candidate for the unstable fixed point between the metallic and localized phases. The subtlety here is that it is not clear whether or not one can continue the well-understood $N > 2$ results to the replica limit. There is

at least one well-known situation (two-dimensional self-avoiding polymers) where one cannot do such a continuation. I am currently studying this question, but do not yet have any conclusive answers.

There are two morals of this section:

1. When the WZW term is present, the model has a non-trivial fixed point.

2. Whether or not the WZW term is present depends on original (microscopic) disordered model considered. For the case of fermions, the coefficient of this term is easily determined.

3. θ term

Another term which can be added to some sigma model actions is called the theta term. This term is inherently non-perturbative: it does not change the beta function derived near the trivial fixed point at all. Nevertheless, it can result in a non-trivial fixed point.

The theta term is best illustrated in the sphere sigma model. As above, I consider field configurations which go to a constant at spatial infinity, so the spatial coordinates are effectively that of a sphere. Since the field takes values on a sphere, the field is therefore a map from the sphere to a sphere. An important characteristic of such maps is that they can have non-trivial topology: they cannot necessarily be continuously deformed to the identity map. This is analogous to what happens when a circle is mapped to a circle (i.e. a rubber band wrapped around a pole): you can do this an integer number of times called the winding number (a negative winding number corresponds to flipping the rubber band upside down). It is the same thing for a sphere: a sphere can be wrapped around a sphere an integer number of times. An example of winding number 1 is the isomorphism from a point on the spatial sphere to the same point on the field sphere. The identity map is winding number 0: it is the map from every point on the spatial sphere to a single point on the field sphere, i.e. $T(x, y) =$constant. In field theory, field configurations with non-zero winding number are usually called instantons. The name comes from viewing one of the directions as time (in our case, one would think of say x as space and y as Euclidean time). Since instanton configurations fall off to a constant at $y = \pm\infty$, the instanton describes a process local in time and hence "instant". In an even fancier modern language, an instanton would be called a -1 brane. Therefore, the field configurations in the sphere sigma model can be classified by an integer n. We can therefore add a term

$$S_\theta = in\theta$$

to the action, where θ is an arbitrary parameter. Since n is an integer, the physics is periodic under shifts of 2π in θ.

Pruisken [20] showed that the replica sigma model describing two-dimensional non-interacting electrons with disorder and a strong transverse magnetic field (which breaks

$\mathcal{T}$) is in the GUE class. This model has instantons with integer winding number, and so allows a θ term. He showed that while the sigma model coupling g is related to the conductivity σ_{xx}, the other parameter θ is related to the Hall conductivity σ_{xy}. He proposed that at $\theta = \pi$, the system has a critical point separating a phase with $\sigma_{xy} = 0$ from $\sigma_{xy} = 1$: the famous (experimentally observed) transition between quantum Hall plateaus. This critical point is stable in g but unstable when θ is taken away from π. While Pruisken's proposed phase diagram is widely believed to be correct, noone has succeeded in deriving any analytic results valid for the replica limit $N \to 0$. The best evidence that Pruisken's phase diagram does apply to GUE class models is indirect, coming from numerical studies of the network model [21]. The network model can be mapped on to a supergroup spin chain [22] whose continuum limit should be described by a supergroup sigma model of the GUE class. Thus ven though the network model is microscopically different from the model of electrons with disorder and transverse magnetic field, it should be in the same universality class. Numerical studies are much easier to do on the network model or on the supergroup spin chain, and the work done is completely consistent with this phase diagram.

Although we do not have any exact results applicable the GUE replica limit, we do a number of exact results for some sigma models at $\theta = \pi$, and I would like to discuss them in this section. These all are in harmony with Pruisken's picture. Although Pruisken did not know this at the time of his proposal (he was reasoning by analogy with some of 't Hooft's work on QCD [23]), the sphere sigma model has the same structure as he proposed for the Hall plateaus. Another way of saying this would be to say the phase structure of the $U(2N)/U(N) \times U(N)$ sigma model is believed to be the same for $N = 1$ and $N \to 0$. While we do not know the exact nature of the non-trivial critical point for $N \to 0$, it is understood well for the sphere, $N = 1$. Namely, Haldane realized when studying the half-integer-spin Heisenberg spin chains that the sphere sigma model at $\theta = \pi$ has a non-trivial fixed point stable in g. This fixed point turns out to be exactly the $SU(2)_1$ WZW model [24]. The argument goes as follows. First one uses Zamolodchikov's c-theorem, which makes precise the notion that as one follows renormalization group flows, the number of degrees of freedom goes down. Zamolodchikov shows that there is a quantity c associated with any two-dimensional unitary field theory such that c must not increase along a flow. At a critical point, c is the central charge of the corresponding conformal field theory [25]. At the trivial fixed point of a sigma model where the manifold is flat, the central charge is the number of coordinates of the manifold (i.e. the number of free bosons which appear in the action (9)). For the sphere, this means that $c = 2$ at the trivial fixed point. The only unitary conformal field theories with $SU(2)$ symmetry and $c < 2$ are $SU(2)_k$ for $k < 4$ (in general, the central charge of $SU(2)_k$ is $3k/(k + 2)$). One can use the techniques of [11] to show that there are relevant operators at these fixed points, and at $k = 2$ or 3, no symmetry of the sphere sigma model prevents these relevant operators from being added to the action [24]. So while it is conceivable that the sphere sigma model with $\theta = \pi$ could flow near to these fixed points, these relevant operators would presumably appear

in the action and cause a flow away. However, there is only one relevant operator (or more precisely, a multiplet corresponding to the field h itself) for the $SU(2)_1$ theory. The sigma model has a discrete symmetry $T \to -T$ when $\theta = \pi$; the winding number n goes to $-n$ under this symmetry, but $\theta = \pi$ and $\theta = -\pi$ are equivalent because of the periodicity in θ. This discrete symmetry of the sigma model presumably turns into the symmetry $h \to -h$ of the WZW model. While the operator $\mathrm{tr}\, h$ is $SU(2)$ invariant, it is not invariant under this discrete symmetry. Therefore, the only possible low-energy fixed point for the sphere sigma model at $\theta = \pi$ is $SU(2)_1$. A variety of arguments involving the spin chain suggest strongly that this does in fact happen [24].

This picture also shows what happens when θ is moved away from π. Here the discrete symmetry is broken but $SU(2)$ is preserved, so one adds $\mathrm{tr}\, h$ to the $SU(2)_1$ action. This is relevant, and in fact is equivalent to the sine-Gordon model (with dimension 1/2 cos term). This is a massive field theory, with no non-trivial low-energy fixed point. The sphere sigma model reproduces exactly Pruisken's phase diagram.

The flow of the sphere sigma model at $\theta = \pi$ to the $SU(2)_1$ WZW model was essentially proven in [26]. This result does not seem to be widely known, so I will review it here. The sphere sigma model is integrable at $\theta = 0$, meaning that there are an infinite number of conserved currents which allow one to find exactly the spectrum of quasiparticles and their scattering matrix in the correspoding $1 + 1$ dimensional field theory. There is evidence that $\theta = \pi$ case is integrable also, so one can assume so and go on to find the quasiparticle S matrix here as well. This is done in [26]. They find that while the quasiparticles for $\theta = 0$ are gapped and form a triplet under the $SU(2)$ symmetry, for $\theta = \pi$ they are gapless, and form $SU(2)$ doublets (left- and right-moving). This is a beautiful example of charge fractionalization: the field T is a triplet under the $SU(2)$ symmetry, but when $\theta = \pi$ the excitations of the system are doublets. They compute the c function, and find that at high energy c indeed is 2 as it should be at the trivial fixed point, while $c = 1$ as it should be at the $SU(2)_1$ low-energy fixed point. As an even more detailed check, the free energy at zero temperature in the presence of a magnetic field was computed for both $\theta = 0$ and π [27]. The results can be expanded in a series around the trivial fixed point. One can identify the ordinary pertubative contributions to this series, and finds that they are the same for $\theta = 0$ and π, even though the particles and S matrices are completely different. This is as it must be: instantons and hence the θ term are a boundary effect and hence cannot be seen in ordinary perturbation theory. One can also identify the non-perturbative contributions to these series, and see that they differ. Far away from the trivial fixed point, non-perturbative contributions can dominate, which is why $\theta = 0$ has no non-trivial fixed point, while $\theta = \pi$ does.

The question now is if similar behavior is found for any other disordered universality classes in two dimensions. In the sigma model language, the question is if any other of the models in Table 2 have instantons and hence allow a θ term. This question has already been answered by mathematicians; for a review accessible to physicists, see [28]. In mathematical language, the question is whether the second homotopy group $\pi_2(G/H)$

is non-trivial. The second homotopy group is just the group of winding numbers of maps from the sphere to G/H, so for the sphere it is the integers. The general answer is that $\pi_2(G/H)$ is the kernel of the embedding of $\pi_1(H)$ into $\pi_1(G)$, where $\pi_1(H)$ is the group of winding numbers for maps of the circle into H. We have seen already that $\pi_1(H)$ is the integers when H is the circle $= U(1) = SO(2)$. The only simple Lie group H for which π_1 is nonzero is $SO(N)$, where $\pi_1(SO(N)) = \mathbf{Z}_2$ for $N \geq 3$ and $\mathbf{Z}$ for N=2. Thus there are models with integer winding number, some with just winding number 0 or 1, and some with no instantons at all. Integer winding number means that θ is continuous and periodic, while a winding number of 0 or 1 means that θ is just 0 or π (just think of θ as being the Fourier partner of n). The results are summarized in the last column of Table 2; the models with integer winding number are labelled as having a Pruisken phase, while those with $\mathbf{Z}_2$ winding number are CII and GSE.

The replica sigma models with integer winding number and continuous θ are believed to behave like the sphere sigma model. In addition to the GUE class, this happens for the $Sp(2N)/U(N)$ sigma models (class C) and the $O(2N)/U(N)$ sigma models (class D). The replica limit of class C should have Pruisken's phase diagram, while in class D it should be modified because of the flip in sign of the beta function: the non-trivial fixed point at $\theta = \pi$ should be unstable in θ and g, and another non-trivial fixed point should appear at some value $g_c, \theta = 0$ (because the metallic phase should not exist at small enough g, i.e. strong enough disorder).

In all three of these universality classes, there is a network model (roughly speaking, one for each type of simple Lie group, $U(N)$, $Sp(2N)$ and $O(2N)$) [21, 29, 30]. In all three cases, numerical results on the network model are consistent with the existence of a non-trivial fixed point as Pruisken predicted. Class D turns out to be a complicated story [18, 32, 13, 30, 31]. It seems that the sigma model does not describe all the physics of this class: because of the existence of domain walls, the complete phase diagram involves more than the two parameters θ and g of the sigma model and Pruisken's phase diagram [13, 31]. The two-dimensional random-bond Ising model belongs to this symmetry class, but is a subspace of this full space. All results support the existence of a non-trivial critical point (or actually, points), but very little is known about detailed properties. On the other hand, the class C model, known as the spin quantum Hall effect (SQHE) [29], is better understood. There is a remarkable exact result mapping certain correlators at the non-trivial fixed point onto a known conformal field theory, that describing percolation [33]. There are no exact results from the replica sigma model point of view, but I have a conjecture to which I will return below.

It is not yet known whether the sigma models in the GUE, C or D classes are integrable. I do have a variety of exact results for the models with $\mathbf{Z}_2$ instantons, namely classes CII and GSE [35]. Without a continuous θ parameter, there does not seem to be any σ_{xy}, so these models do not have the full structure of the above three classes. However, these two models still have a non-trivial fixed point when $\theta = \pi$, and for this reason I believe they provide strong support for Pruisken's picture.

Basically, my results generalize those of [26] to this much more general case. This is

important because to have any hope of being able to take the replica limit, one needs a solution for any N. For class CII, the sigma models are on the space $U(N)/O(N)$. This sigma model has action (9) with T a symmetric, unitary matrix. I find that when $\theta = 0$, these sigma models are the $U(N)$ generalization of the sphere sigma model. When $\theta = 0$, the model has a gap, with the spectrum consisting of massive particles in the symmetric representation of $SU(N)$ (plus bound states in more general representations). When $\theta = \pi$, the spectrum consists of gapless quasiparticles which are in the fundamental representations (vector, antisymmetric tensor,...) of $SU(N)$. The non-trival low-energy fixed point when $\theta = \pi$ corresponds to $SU(N)_1 \times U(1)$. Thus we see that at least for $N > 0$, the replica sigma models in class CII with $\theta = \pi$ have exactly the same fixed point as those in class AIII when a $k = 1$ WZW term is present! The effect of having $\theta = \pi$ is as discussed before: the density of states changes its behavior near special values of the coupling.

The results for the GSE class are similar. This model is the $O(2N)/O(N) \times O(N)$ sigma model, which has action (9) with T a symmetric, real and orthogonal matrix. When $\theta = 0$, the model is gapped. When $\theta = \pi$, the model is gapless with a non-trivial stable fixed point corresponding to the $O(2N)_1$ WZW model. This sigma model proves to be the $O(2N)$ generalization of the sphere sigma model. The $O(2N)_1$ model turns out to be $2N$ free Majorana fermions. The word "free" is slightly deceptive, because just as in the 2d Ising model, one can study correlators of the magnetization or "twist" operator, which are highly non-trivial. Because of the changes in the beta function at N=1, it is not clear yet whether these results can be continued to the replica limit; I am currently studying this. However, again it proves that the idea of a non-trivial critical point at $\theta = \pi$ is not a fluke of the sphere sigma model, and is true for any N.

The cases with $\mathbf{Z}_2$ instantons are very similar to the WZW cases: θ is not a tunable parameter. In fact, I believe it is fixed uniquely by the underlying disordered system. Indeed, the expression for the winding number as an integral over the fields is precisely of the form of the WZW term. The deep connection between anomalies and theta terms was discussed in [36].

There are three models with a Pruisken phase, roughly corresponding to the three kinds of Lie groups $U(N)$, $Sp(2N)$ and $O(2N)$. There are only two models with $\mathbf{Z}_2$ instantons, roughly corresponding to $U(N)$ and $O(2N)$ type, flowing to $U(N)_1$ and $O(2N)_1$ when $\theta = \pi$. It is logical to ask if there a sigma model with $Sp(2N)$ symmetry resembling the latter two. From the replica point of view, it is clearly the sigma model in class C, namely $Sp(2N)/U(N)$. The reason I view this as analogous is that the sigma model has action (9), where T is unitary and symmetric like the other two, but with the additional restriction that $\mathrm{tr}(JT) = 0$, where J is the $2N \times 2N$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $N \times N$ identity matrix. One can presumably obtain the $Sp(2N)/U(N)$ sigma model from the $SU(2N)/SO(2N)$ model by pertubing by something like $\lambda \int (\mathrm{tr}\, JT)^2$.

This breaks the global $SU(2N)$ symmetry to $Sp(2N)$. The question is if when $\theta = \pi$, there remains a non-trivial critical point after perturbing. For $N = 1$, the two sigma models are the same ($Sp(2) = SU(2)$) so obviously the fixed points are the same. For $N > 1$, any non-trivial fixed point in $Sp(2N)/U(N)$ should be a perturbation of the $\theta = \pi$ fixed point of $SU(2N)/SO(N)$, namely $SU(2N)_1$, which has central charge $2N-2$. If there is a non-trivial critical point of $Sp(2N)/U(N)$ when $\theta = \pi$, it must have central charge less than $2N - 2$, which leaves only $Sp(2N)_1$. Obviously, this is not a proof there is such a point, and moreover, it does not say if this behavior can be continued to $N \to 0$. Nevertheless, if this correspondence holds, it predicts that in class C the density of states $\rho(E) \propto E^{1/7}$ [12, 4]. This agrees with the exact result of [33] derived from the map onto percolation. It also predicts that there is another relevant operator of positive dimension $5/4$, which is thermal operator in percolation and the operator corresponding to moving off the critical point in the network model. These numbers also can be found from the analogous supersymmetric approach [34]. While I think the agreement of dimensions is not a coincidence, this hardly proves that the non-trivial fixed point in class C is $Sp(2N)_1$. It would be much more convincing if a correlator could be computed and shown to be equivalent correlator in percolation.

So this section has two morals virtually identical to those in the last:

1. All the available evidence suggests that when $\theta = \pi$, there is a non-trivial fixed point, in support of Pruisken's scenario.
2. For models with $\mathbf{Z}_2$ instantons, the underlying disordered system should determine if $\theta = \pi$ or 0.

I want to add a third moral:

3. Relevant operators may not always be relevant.

What I mean by the last is best illustrated by an example, following [24]. Consider the principal chiral model on $SU(2)$ with the action (9), with the field T taking values in $SU(2)$. Now add a $k = 1$ WZW term, (10) with $h = T$. As noted before, this causes a flow to the stable fixed point $SU(2)_1$. Say in addition to adding the WZW term, I also add a term $\lambda(\mathrm{tr}\, T)^2$. Around the trivial fixed point, this is a relevant perturbation, breaking the chiral symmetry but not the diagonal $SU(2)$. One might think it wrecks the flow to the $SU(2)_1$ (chirally-invariant) fixed point. However, it does not necessarily. An $SU(2)$ matrix T can be rewritten as

$$\begin{pmatrix} n_0 + in_1 & n_2 + in_3 \\ -n_2 + in_3 & n_0 - in_1 \end{pmatrix}$$

where the otherwise-free parameters must satisfy $(n_0)^2 + (n_1)^2 + (n_2)^2 + (n_3)^2 = 1$. The chiral-symmetry-breaking perturbation is $\lambda(n_0)^2$. For λ large, its effect is to force $n_0 = 0$, leaving $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$. This is the sphere: this relevant pertubation turns the principal chiral model into the sphere sigma model. The WZW term turns into the theta

term of the sigma model, with $\theta = k\pi$. The result discussed above shows that if k is an odd integer, the presence of both the WZW term and the chiral-symmetry-breaking perturbation does not result in a massive theory: one ends up at the $SU(2)_1$ fixed point! For $k = 1$, one ends up exactly where one would have otherwise, although the flow does reach the $SU(2)_1$ fixed point from a different (chirally non-invariant) direction. In fact one can see directly at the $SU(2)_1$ fixed point that all fields $T_{\alpha\beta}T_{\gamma\delta}$ operator is irrelevant there [11]. However, I think this is a useful moral for the situation with an unknown low-energy fixed point: just because there is a relevant operator at the trivial fixed point doesn't necessarily mean it will always be relevant.

I have discussed how a non-trivial fixed point can appear in a two-dimensional replica sigma model. These are summarized in Table 2. Every universality class save one has at least one kind of possible non-trivial critical point. Ironically, the only one that does not is Anderson's original problem of free electrons with disorder!

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